New non-arithmetic lattices in $\text{SU}(2, 1)$

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Introduction: lattices in real semisimple Lie groups

Main results

Complex Hyperbolic Space and Isometries

Mostow’s lattices

Configuration space of symmetric complex reflection triangle groups

Sporadic groups

Discreteness and Fundamental Domains

Commensurability classes

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Appendix: statement of Poincaré polyhedron theorem
$G$: real semisimple Lie group.

$\Gamma \subset G$ is a lattice if $\Gamma \backslash G$ has finite volume (Haar measure).

$\Gamma$ is cocompact (C) if $\Gamma \backslash G$ is compact, non-cocompact (NC) otherwise.

**Question 1:** Do there exist lattices in $G$?

Yes (Borel–Harish-Chandra), the so-called arithmetic lattices (e.g. $\text{SL}(n,\mathbb{Z}) \subset \text{SL}(n,\mathbb{R})$, $\text{SL}(n,\mathbb{Z}[i])$ or $\text{SL}(n,\mathbb{Z}[\omega]) \subset \text{SL}(n,\mathbb{C})$).

Infinitely many, C and NC.

**Question 2:** Do there exist any other lattices in $G$? If yes, how many?

▶ In $\text{SL}(2,\mathbb{R})$: yes, lots. (Arithmetic=small, Teichmüller=BIG).

▶ No, if R-Rank($G$) $\geq 2$ (Margulis).
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**Minor open question:** Do there exist infinitely many (non-commensurable) non-arithmetic lattices in \( \text{SU}(2, 1) \)?
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Appendix: statement of Poincaré polyhedron theorem
We produced in earlier work with J. Parker an infinite list of subgroups of $\text{SU}(2, 1)$ which we called sporadic (complex hyperbolic symmetric triangle) groups, which are candidates for being new non-arithmetic lattices.
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**Theorem (P)**

All but one of the sporadic groups are non-arithmetic. None are commensurable to Picard or Mostow lattices (with a small list of possible exceptions).

**Conjecture (DPP)**

At least 11 of the sporadic groups are lattices, 4 C and 7 NC. We also conjecture that almost all others are non-discrete (for 3 groups we don't conjecture anything). So far the conjecture has been established in most cases.
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We constructed Dirichlet domains (numerically, with M. Deraux's program) for lots of these groups, which led us to:

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*The six groups $\Gamma(2\pi/p, \overline{\sigma_4})$ ($p = 3, 4, 5, 6, 8, 12$) are lattices.*

Note that: one of these ($p = 3$) is the arithmetic one, the others are all non-arithmetic and new (3 C and 2 NC). The proof is by construction of a fundamental domain in $H^2_\mathbb{C}$, so we also get presentations, volumes,...
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Then $H^n_\mathbb{C} := \pi(V^-) \subset \mathbb{CP}^n$, with distance $d$ (Bergman metric) given by:

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From this formula it is clear that $\text{PU}(n, 1)$ acts by isometries on $H^n_{\mathbb{C}}$ (where $\text{U}(n, 1) < \text{GL}(n + 1, \mathbb{C})$ is the subgroup preserving $\langle \cdot, \cdot \rangle$).
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In fact: $\text{Isom}^{+}(H_{\mathbb{C}}^{n}) = PU(n, 1)$, and $\text{Isom}(H_{\mathbb{C}}^{n}) = PU(n, 1) \rtimes \mathbb{Z}/2$ (complex conjugation).
Totally geodesic subspaces: The only totally geodesic subspaces of $H^n_C$ are the projective images of complex linear subspaces (copies of $H^k_C \subset H^n_C$) and of totally real subspaces (copies of $H^k_R \subset H^n_C$).
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**Definition:** A complex reflection is an elliptic isometry $g$ with $\text{Fix}(g)$ of (complex!) codimension 1.

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(On a remarkable class of polyhedra in complex hyperbolic space, PJM 1980)

Notation: $\Gamma(\mu, t) < SU(2, 1)$, where $\mu = 3, 4$ or $5$ and $t$ is a real parameter. 

The $\Gamma(\mu, t)$ are symmetric complex reflection triangle groups, i.e.:

$\Gamma = \langle R_1, R_2, R_3 \rangle$ where each $R_i$ is a complex reflection of order $\mu$.

Symmetric means that there exists an isometry $J$ of order $3$ such that $JR_iJ^{-1} = R_i+1$, or equivalently $J(L_i) = L_i+1$ where $L_i = \text{Fix}(R_i)$.

Moreover Mostow imposes the braid relation $R_iR_jR_i = R_jR_iR_j$.

Facts:

For fixed $\mu$ there is a 1-dimensional family of such groups (hence the $t$).
Only finitely many of the $\Gamma(\mu, t)$ are discrete; the discrete ones are lattices.
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(On a remarkable class of polyhedra in complex hyperbolic space, PJM 1980)

Notation: $\Gamma(p, t) < \text{SU}(2, 1)$, where $p = 3, 4$ or $5$ and $t$ is a real parameter.

The $\Gamma(p, t)$ are symmetric complex reflection triangle groups, i.e.:
- $\Gamma = \langle R_1, R_2, R_3 \rangle$ where each $R_i$ is a complex reflection of order $p$.
- $\text{symmetric}$ means that there exists an isometry $J$ of order 3 such that $JR_iJ^{-1} = R_{i+1}$, or equivalently $J(L_i) = L_{i+1}$ where $L_i = \text{Fix}(R_i)$.

Moreover Mostow imposes the braid relation $R_iR_jR_i = R_jR_iR_j$.

Facts:
- For fixed $p$ there is a 1-dimensional family of such groups (hence the $t$).
- Only finitely many of the $\Gamma(p, t)$ are discrete; the discrete ones are lattices.
Introduction: lattices in real semisimple Lie groups

Main results

Complex Hyperbolic Space and Isometries

Mostow’s lattices

Configuration space of symmetric complex reflection triangle groups

Sporadic groups

Discreteness and Fundamental Domains

Commensurability classes

Non-arithmeticity

Appendix: statement of Poincaré polyhedron theorem
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In concrete terms, we use either:

- $\tau := \text{Tr} R_1 J$ (good for arithmetic), or
- The angle pair $\{\theta_1, \theta_2\}$ of $R_1 J$ when elliptic (good for geometry).
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- $\tau := \text{Tr} R_1 J$ (good for arithmetic), or
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**Notation:** We denote $\Gamma(2\pi/p, \tau) = \langle R_1, J \rangle$, where $R_1$ is a complex reflection through angle $2\pi/p$, $J$ a regular elliptic isometry of order 3, and $\tau := \text{Tr} R_1 J$. 
Introduction: lattices in real semisimple Lie groups

Main results

Complex Hyperbolic Space and Isometries

Mostow’s lattices

Configuration space of symmetric complex reflection triangle groups

Sporadic groups

Discreteness and Fundamental Domains

Commensurability classes

Non-arithmeticity

Appendix: statement of Poincaré polyhedron theorem
Theorem (Parker-P.)

Let $R_1$ be a complex reflection of order $p$ and $J$ a regular elliptic isometry of order 3 in $\text{PU}(2,1)$. Suppose that $R_1J$ and $R_1R_2 = R_1JR_1J^{-1}$ are elliptic or parabolic. If the group $\Gamma = \langle R_1, J \rangle$ is discrete then one of the following is true:

- $\Gamma$ is one of Mostow’s lattices.
- $\Gamma$ is a subgroup of one of Mostow’s lattices.
- $\Gamma$ is one of the sporadic groups listed below.
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Mostow’s lattices correspond to $\tau = e^{i\phi}$ for some angle $\phi$; subgroups of Mostow’s lattices to $\tau = e^{2i\phi} + e^{-i\phi}$ for some angle $\phi$, and sporadic groups are those for which $\tau$ takes one of the 18 values $\{\sigma_1, \overline{\sigma_1}, ..., \sigma_9, \overline{\sigma_9}\}$ where the $\sigma_i$ are given in the following list:
\[\sigma_1 := e^{i\pi/3} + e^{-i\pi/6} \cdot 2 \cos(\pi/4)\]
\[\sigma_3 := e^{i\pi/3} + e^{-i\pi/6} \cdot 2 \cos(2\pi/5)\]
\[\sigma_5 := e^{2\pi i/9} + e^{-i\pi/9} \cdot 2 \cos(2\pi/5)\]
\[\sigma_7 := e^{2\pi i/9} + e^{-i\pi/9} \cdot 2 \cos(2\pi/7)\]
\[\sigma_9 := e^{2\pi i/9} + e^{-i\pi/9} \cdot 2 \cos(6\pi/7).\]

\[\sigma_2 := e^{i\pi/3} + e^{-i\pi/6} \cdot 2 \cos(\pi/5)\]
\[\sigma_4 := e^{2\pi i/7} + e^{4\pi i/7} + e^{8\pi i/7}\]
\[\sigma_6 := e^{2\pi i/9} + e^{-i\pi/9} \cdot 2 \cos(4\pi/5)\]
\[\sigma_8 := e^{2\pi i/9} + e^{-i\pi/9} \cdot 2 \cos(4\pi/7)\]
\[ \sigma_1 := e^{i\pi/3} + e^{-i\pi/6} 2 \cos(\pi/4) \quad \sigma_2 := e^{i\pi/3} + e^{-i\pi/6} 2 \cos(\pi/5) \]
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\[ \sigma_9 := e^{2\pi i/9} + e^{-i\pi/9} 2 \cos(6\pi/7). \]

Therefore, for each value of \( p \geq 3 \), we have a finite number of groups to study, the \( \Gamma(2\pi/p, \sigma_i) \) and \( \Gamma(2\pi/p, \sigma_i) \) which are hyperbolic (i.e. preserve a form of signature (2,1)).
Introduction: lattices in real semisimple Lie groups

Main results

Complex Hyperbolic Space and Isometries

Mostow’s lattices

Configuration space of symmetric complex reflection triangle groups

Sporadic groups

Discreteness and Fundamental Domains

Commensurability classes

Non-arithmeticity

Appendix: statement of Poincaré polyhedron theorem
Figure: A view of the domain $E$ for $p = 12$
Theorem (DPP)

Let $p \geq 3$, $R_1 \in SU(2, 1)$ be a complex reflection through angle $2\pi/p$ and $J \in SU(2, 1)$ be a regular elliptic map of order 3. Suppose that $\tau = \text{Tr}(R_1 J) = \overline{\sigma_4} = -(1 + i\sqrt{7})/2$. 

Define $c = 2p/(p-4)$ and $d = 2p/(p-6)$. The group $\langle R_1, J \rangle$ is a lattice whenever $c$ and $d$ are both integers, possibly infinity, that is when $p = 3, 4, 5, 6, 8, 12$. Moreover, writing $R_2 = J R_1 J^{-1}$ and $R_3 = J^{-1} R_1 J$, this group has presentation $\langle R_1, R_2, R_3, J \mid R_1^p = J^3 = (R_1 J)^7 = \text{id}, R_2 = J R_1 J^{-1}, R_3 = J^{-1} R_1 J, (R_1 R_2)^2 = (R_2 R_1)^2, (R_1 R_2)^2 c = (R_1 R_2 R_3 R_1^{-1})^3 d = \text{id} \rangle$. 


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Moreover, writing \( R_2 = JR_1J^{-1} \) and \( R_3 = JR_2J^{-1} = J^{-1}R_1J \), this group has presentation

\[
\left\langle R_1, R_2, R_3, J \mid \begin{align*}
R_1^p &= J^3 = (R_1J)^7 = id, \\
R_2 &= JR_1J^{-1}, \quad R_3 = J^{-1}R_1J, \\
(R_1R_2)^2 &= (R_2R_1)^2, \\
(R_1R_2)^{2c} &= (R_1R_2R_3R_2^{-1})^{3d} = id
\end{align*} \right\rangle.
\]
Bisectors:

Given 2 distinct points $p_1, p_2 \in H^n_C$, the *bisector* equidistant from $p_1, p_2$ is:

$$B(p_1, p_2) = \{ p \in H^n_C | d(p, p_1) = d(p, p_2) \}.$$
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The *complex spine* $\Sigma$ of $B = B(p_1, p_2)$ is the complex line spanned by $p_1, p_2$; the *real spine* $\sigma$ of $B$ is the real geodesic $B \cap \Sigma$. 

Bisectors are not totally geodesic, but they admit 2 foliations by totally geodesic subspaces, called its *slices* and *meridians*:

- Proposition 1. (Mostow) $B = \pi^{-1}(\Sigma)(\sigma)$.
- Proposition 2. (Goldman) $B$ is the union of all real planes containing $\sigma$.
- Proposition 3. (Goldman) Given 2 distinct points $p, q \in B$, the geodesic $(pq)$ is contained in $B$ iff $p, q$ are in a common slice or meridian.

The intersection between a bisector $B$ and a geodesic $g \not\subset B$ may contain 0, 1 or 2 points.
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Description of the domains $D$ and $E$:

We construct 2 related polyhedra in $\mathbb{H}^2_\mathbb{C}$. $D$ will be a fundamental domain for the lattice $\Gamma$, and $E$ will be a fundamental domain for the action of $\Gamma$ modulo $\langle P \rangle$, where $P = R_1J$ has order 7.
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$E$ is then defined as the intersection of the 28 half-spaces bounded by $P^k(R^\pm)$, $P^k(S^\pm)$ ($k = 0, \ldots, 6$) and containing $O_P$, the isolated fixed point of $P$. 
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Proposition

$E$ is cell-homeomorphic to a convex polytope in $\mathbb{R}^4$ (with some vertices removed when $\Gamma$ is NC).
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**Theorem (Giraud, 1934)**

If $B_1$ and $B_2$ are 2 coequidistant bisectors, then $B_1 \cap B_2$ is a (non-totally geodesic) smooth disk. Moreover, there exists a unique bisector $B_3 \neq B_1, B_2$ containing it.

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Bad projections of Giraud disks:
Introduction: lattices in real semisimple Lie groups

Main results

Complex Hyperbolic Space and Isometries

Mostow’s lattices

Configuration space of symmetric complex reflection triangle groups

Sporadic groups

Discreteness and Fundamental Domains

Commensurability classes

Non-arithmeticity

Appendix: statement of Poincaré polyhedron theorem
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For $\Gamma = \Gamma(2\pi/p, \bar{\sigma}_4)$, $\mathbb{Q}[\text{TrAd}\Gamma] = \mathbb{Q}[\cos \frac{2\pi}{p}, \sqrt{7} \sin \frac{2\pi}{p}]$. 
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Corollary

(1) The 6 groups $\Gamma = \Gamma(2\pi/p, \overline{\sigma_4})$ with $p = 3, 4, 5, 6, 8, 12$ lie in different commensurability classes.

(2) The 6 groups $\Gamma = \Gamma(2\pi/p, \overline{\sigma_4})$ with $p = 3, 4, 5, 6, 8, 12$ are not commensurable to any Deligne-Mostow lattice.
Introduction: lattices in real semisimple Lie groups

Main results

Complex Hyperbolic Space and Isometries

Mostow’s lattices

Configuration space of symmetric complex reflection triangle groups

Sporadic groups

Discreteness and Fundamental Domains

Commensurability classes

**Non-arithmeticity**

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Proposition (Vinberg, Mostow)

Let $E$ be a purely imaginary quadratic extension of a totally real field $F$, and $H$ a Hermitian form of signature $(2,1)$ defined over $E$.

1. $\text{SU}(H;O_E)$ is a lattice in $\text{SU}(H)$ if and only if for all $\phi \in \text{Gal}(F)$ not inducing the identity on $F$, the form $\phi H$ is definite. In that case, $\text{SU}(H;O_E)$ is an arithmetic lattice.

2. Suppose $\Gamma \subset \text{SU}(H;O_E)$ is a lattice. Then $\Gamma$ is arithmetic if and only if for all $\phi \in \text{Gal}(F)$ not inducing the identity on $F$, the form $\phi H$ is definite.

Note that when the group $\Gamma$ as in the Proposition is non-arithmetic, it necessarily has infinite index in $\text{SU}(H;O_K)$ (which is non-discrete in $\text{SU}(H)$).
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2. Suppose $\Gamma \subset \text{SU}(H;\mathcal{O}_E)$ is a lattice. Then $\Gamma$ is arithmetic if and only if for all $\varphi \in \text{Gal}(F)$ not inducing the identity on $F$, the form $\varphi H$ is definite.
Proposition (Vinberg, Mostow)

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Note that when the group $\Gamma$ as in the Proposition is non-arithmetic, it necessarily has infinite index in $SU(H, \mathcal{O}_K)$ (which is non-discrete in $SU(H)$).
Introduction: lattices in real semisimple Lie groups

Main results

Complex Hyperbolic Space and Isometries

Mostow’s lattices

Configuration space of symmetric complex reflection triangle groups

Sporadic groups

Discreteness and Fundamental Domains

Commensurability classes

Non-arithmeticity

Appendix: statement of Poincaré polyhedron theorem
**Definition:** A Poincaré polyhedron is a smooth polyhedron $D$ in $X$ with codimension one faces $T_i$ such that

1. The codimension one faces are paired by a set $\Delta$ of isometries of $X$ which respect the cell structure (the side-pairing transformations). We assume that if $\gamma \in \Delta$ then $\gamma^{-1} \in \Delta$.
2. For every $\gamma_{ij} \in \Delta$ such that $T_i = \gamma_{ij} T_j$ then $\gamma_{ij} D \cap D = T_i$. 
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**Remark:** If $T_i = T_j$, that is if a side-pairing maps one side to itself then we impose, moreover, that $\gamma_{ij}$ be of order two and call it a reflection. We refer to the relation $\gamma_{ij}^2 = 1$ as a reflection relation.
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**Cycles:** Let $T_1$ be an $(n - 1)$-face and $F_1$ be an $(n - 2)$-face contained in $T_1$. Let $T'_1$ be the other $(n - 1)$-face containing $F_1$. Let $T_2$ be the $(n - 1)$-face paired to $T'_1$ by $g_1 \in \Delta$ and $F_2 = g_1(F_1)$. Again, there exists only one $(n - 1)$-face containing $F_2$ which we call $T'_2$. We define recursively $g_i$ and $F_i$, so that $g_{i-1} \circ \cdots \circ g_1(F_1) = F_i$. 
Definition: Cyclic is the condition that for each pair \((F_1, T_1)\) (an \((n - 2)\)-face contained in an \((n - 1)\)-face), there exists \(r \geq 1\) such that, in the construction above, \(g_r \circ \cdots g_1(T_1) = T_1\) and \(g_r \circ \cdots g_1\) restricted to \(F_1\) is the identity.
**Definition:** *Cyclic* is the condition that for each pair \((F_1, T_1)\) (an \((n - 2)\)-face contained in an \((n - 1)\)-face), there exists \(r \geq 1\) such that, in the construction above, \(g_r \circ \cdots \circ g_1(T_1) = T_1\) and \(g_r \circ \cdots \circ g_1\) restricted to \(F_1\) is the identity. Moreover, calling \(g = g_r \circ \cdots \circ g_1\), there exists a positive integer \(m\) such that
\[
g_1^{-1}(P) \cup (g_2 \circ g_1)^{-1}(P) \cup \cdots \cup g^{-1}(P) \cup (g_1 \circ g)^{-1}(P) \cup (g_2 \circ g_1 \circ g)^{-1}(P) \cup \cdots \cup (g^m)^{-1}(P)
\]
is a cover of a closed neighborhood of the interior of \(F_1\) by polyhedra with disjoint interiors.

The relation \(g^m = (g_r \circ \cdots \circ g_1)^m = \text{Id}\) is called a *cycle relation*.

**Theorem**
Let \(D\) be a compact Poincaré polyhedron in \(H^n\) with side-pairing transformations \(\Delta\) satisfying condition *Cyclic*. Let \(\Gamma\) be the group generated by \(\Delta\). Then \(\Gamma\) is a discrete subgroup of \(\text{Isom}(H^n)\), \(D\) is a fundamental domain for \(\Gamma\) and \(\Gamma\) has presentation:
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\Gamma = \langle \Delta | \text{cycle relations, reflection relations} \rangle
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